Asymptotic Method for Option Pricing with Stochastic Volatility and Markovian Regimes Shifts

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Outline

Financial Mathematics

There are many topics in Finantial Mathematics such as

1. Pricing for Financial Derivatives (how to pay)
   - Option Price
   - Bond Price
2. Portfolio Problems (how to invest)
   - Mutual Fund
   - Investment
3. Risk Problems (how to save)
   - Credit
   - Morgages

The Basic Concepts of Call Option

European Call Option's Payoff Function : $[S_T - K]^+$

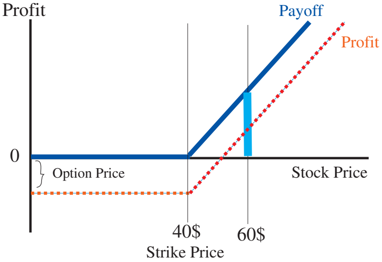


Figure: Graphical interpretation of the payoffs and profits generated by Call Option as buyer

Example
If we buy SNEDH(Call Option), we agreed to buy SONY stock at 40$ from seller of the option at April 19, 2008

Black-Scholes Option Pricing Model

Black,F., Scholes, M. (1973) and Merton, R. (1973)
Merton and Scholes received the 1997 Nobel Prize in Economics …

Let consider asset S_t with riskless asset β_t

$$\begin{aligned} dS_t &= \mu S_t dt + \bar{\sigma} S_t dB_t \qquad \text{GBM} \\ d\beta_t &= r dt \end{aligned} \tag{1}$$

European Call Option's Payoff Function : $[S_T - K]^+$
then,

$$C_t = e^{-r\tau}\mathbb{E}^\star[(S_T^\star - K, 0)^+] \tag{2}$$

with Risk Neutral Probability Measure.

Black-Scholes Option Pricing Model (Continued)

By discounted The Feynmann-Kac formula,

$$\begin{aligned} \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + r\big(x\frac{\partial P}{\partial x} - P\big) &= 0 \\ P(T, x) &= h(x) \end{aligned} \tag{3}$$

The Solution is

$$C_t = x\mathbf{\Phi}(d_+(\tau, x)) - Ke^{-r(\tau)}\mathbf{\Phi}(d_-(\tau, x)) \tag{4}$$

where

$$\begin{aligned} d_\pm(\tau, x) &= \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{x}{K} + (r \pm \frac{1}{2}\sigma^2)\tau\right] \\ \mathbf{\Phi}(z) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-y^2/2}dy \end{aligned} \tag{5}$$

Black-Scholes Option Pricing Model (Continued)

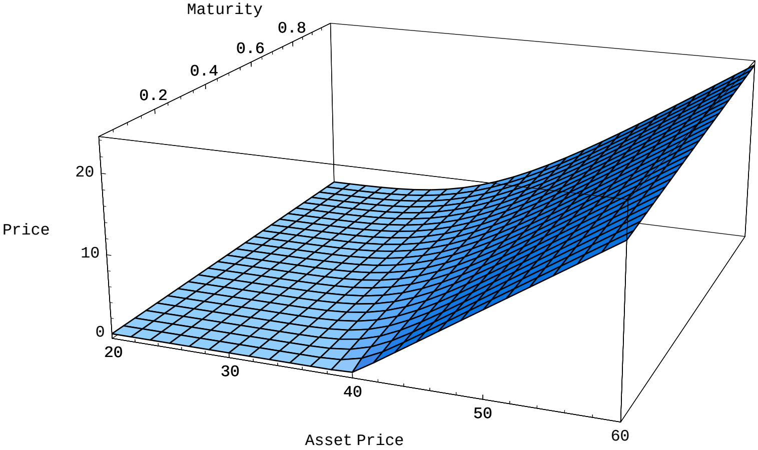
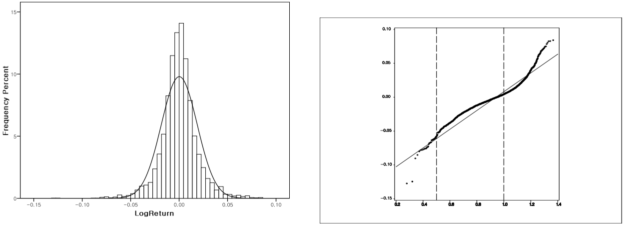


Figure: Numerical solution for the Black-Scholes European option pricing function

Fat-tailness of Log Return

The assumption : Log Return is Normal Distribution



(a) Fat-fail in log-return

(b) Q-Q plot

Figure: Empirical evidence in KOSPI200

Black Monday in 1987

October 19, 1987
Dow Jones Index dropped by 508 points to 1739 (22.6 %) . . .

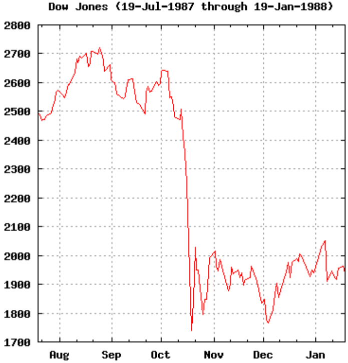


Figure: Black Monday of Dow Jones in 1987

Historical Overview for Correcting

- Constant Volatility - Black,Scholes(1973)
- Historical Volatility (GARCH Model) - Johnson(1987), Bollerslev(1988), Schwert(1989)
- Stochastic Volatility - Hull-White(1987), Scott(1987), Wiggins(1987), Stein-Stein(1991), Ball-Roma(1994), Heston(1993)
- Stochastic Volatility With Perturbation - Fouque, J.P., Papanicolaou, G., and Sircar, R. (1999)

Stochastic Volatility Model

Fouque, J.P., Papanicolaou, G., and Sircar, R. (1999)

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma_t X_t dW_t \\ \sigma_t &= f(Y_t). \\ dY_t &= \alpha(m - Y_t)dt + \beta d\hat{Z}_t \end{aligned} \tag{6}$$

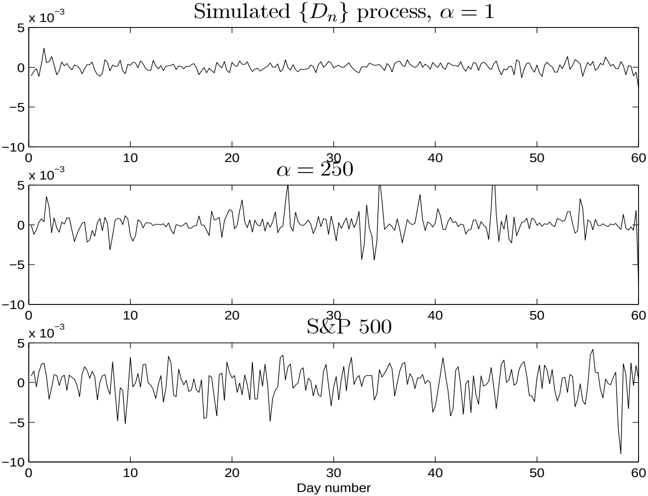
the corrected price is given explicitly by

$$\begin{aligned} P^{\varepsilon} &= P_0 - (T-t)\left(V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x^3 \frac{\partial^3}{\partial x^3}\right) P_0 \\ &= P_0 - (T-t)\sqrt{\varepsilon}\mathcal{H}(t,x)P_0 \end{aligned} \tag{7}$$

where $P_0 = P_{BS}(\bar{\sigma})$,the Black-Scholes price with constant volatility $\bar{\sigma} = \langle f(y) \rangle$

Simulation of Stochastic Volatility Model

Fouque, J.P., Papanicolaou, G., and Sircar, R. (2000)



Multi-Scale Stochastic Volatility Model

Fouque, J.P., Papanicolaou, G., Sircar, R.and Solna,K. (2003)

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma_t X_t dB_t^{(1)} \\ \sigma_t &= f(Y_t, Z_t). \\ dY_t &= \alpha_1(m - Y_t)dt + \beta_1 d\hat{B}_t^{(2)} \\ dZ_t &= \alpha_2(m - Z_t)dt + \beta_2 d\hat{B}_t^{(3)} \end{aligned} \tag{8}$$

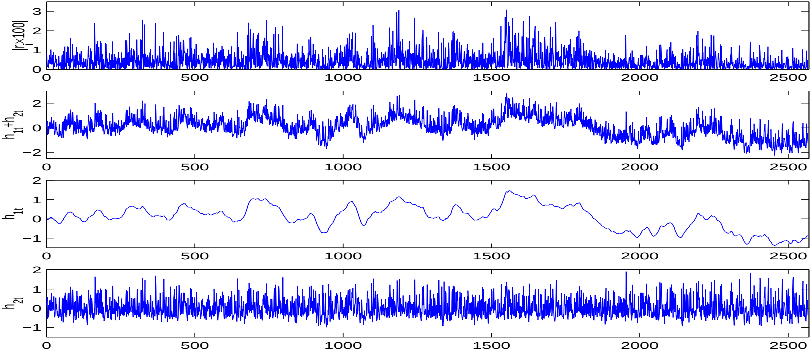
the corrected price is given explicitly by

$$P^{\varepsilon,\delta} = P_0 + (T-t)\Big(-\sqrt{\varepsilon}\mathcal{H}(t,x) + \frac{\sqrt{\delta}}{2}\langle \mathcal{M}_1 \rangle \Big) P_0 \tag{9}$$

where $P_0 = P_{BS}(\bar{\sigma})$,the Black-Scholes price with constant volatility $\bar{\sigma} = \langle f(y) \rangle$

Simulation of Multiscale Stochastic Volatility

Molina G., Han, C.H., and Fouque J.P.(2004)



Stochastic Volatility With Markovian Regimes Shifts

New Model !!

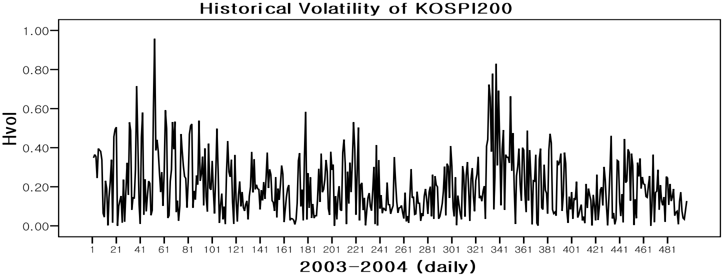
$$\begin{aligned} dX_t =& \mu X_t dt + \sigma_t X_t dB_t^{(1)} \\ \sigma_t =& f(Y_t, Z_t). \\ dY_t =& \alpha(m - Y_t)dt + \beta d\hat{B}_t^{(2)} \\ Z_t :=& \text{Markovian Jump Process} \end{aligned} \tag{10}$$

European Call Option's Payoff Function : $[S_T - K]^+$
then,

$$C_t = e^{-r\tau}\mathbb{E}^{\star}[(S_T^{\star} - K, 0)^+] \tag{11}$$

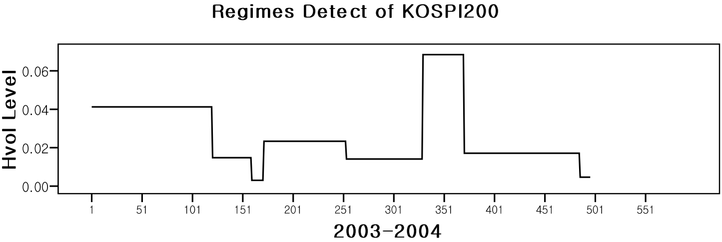
with Risk Neutral Probability Measure.

Motivation of SVRS Model



Motivation of Regimes

by Sequential F-test - Rodionov R.(2001)



Regime Shifts in Markovian Jump process

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in terms of the parameter δ, we assume that the resime switching followed the process Z_t^δ where the number of jump N_t before time t is a poisson process with intensity $\lambda = \sqrt{\delta}$:

$$\mathbb{P}\big(N_t = k\big) = \frac{(\sqrt{\delta}t)^k}{k!} e^{-\sqrt{\delta}t} \tag{12}$$

for nonnegative integers k. The density function of jump size is uniformly distrubution :

$$p(u) = \frac{1}{2} \mathbf{1}_{(-1,1)}(u) \tag{13}$$

Regime Shifts in Markovian Jump process

Definition

The infinitesimal generator of process is defined by :

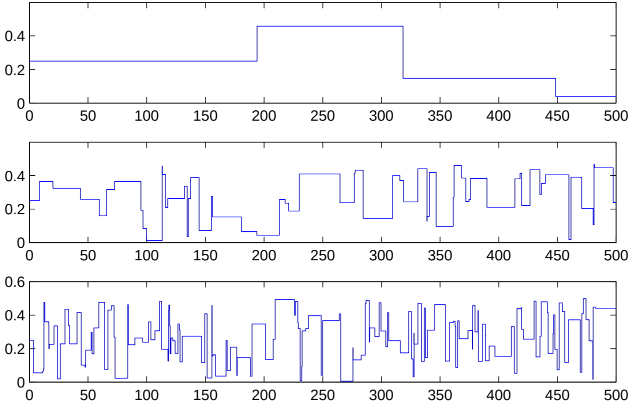
$$\mathcal{A}g(y, Z_t) := \lim_{t \downarrow 0} \frac{\mathbb{E}\big[g(y, Z_t)\big] - g(y, z)}{t} \tag{14}$$

Lemma

Infinitesimal generator of Markovian jump process is

$$\mathcal{A}g(y, Z_t) = \sqrt{\delta} \int \big(g(y, u) - g(y, z_0)\big) p(u) du \tag{15}$$

Simulated sample paths of Makovian Jump Process



Fast Scale Stochastic Volatility

We control the OU process with ε

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{B}_t^{(2)}$$

by $\varepsilon = \frac{1}{\alpha}$ and $\beta = \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}$ then

$$dY_t^{\varepsilon} = \frac{1}{\varepsilon}(m - Y_t^{\varepsilon})dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}d\hat{B}_t^{(2)} \tag{16}$$

Risk Neutral Model

$$\begin{aligned} dX_t^{\varepsilon,\delta} =& rX_t^{\varepsilon,\delta}dt + f(Y_t^\varepsilon, Z_t^\delta)X_t^{\varepsilon,\delta}dB_t^{(1)\star} && (17)\\ dY_t^\varepsilon =& \Big[\frac{1}{\varepsilon}(m - Y_t^\varepsilon) + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon)\Big]dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}d\hat{B}_t^{(2)\star} && (18)\\ Z_t^\delta =& Z_{t/\sqrt{\delta}} && (19) \end{aligned}$$

with

$$\hat{B}_t^{(2)\star} = \rho B_t^{(1)\star} + \sqrt{1-\rho^2}B_t^{(2)\star} \tag{20}$$

then risk-neutral price of European call option is,

$$P^{\varepsilon,\delta}(t,x,y,z) = \mathbb{E}^{\star(\gamma)}\left\{e^{-r(T-t)}h(X_T^{\varepsilon,\delta})|X_t^{\varepsilon,\delta} = x, Y_t^\varepsilon = y, Z_t^\delta = z\right\} \tag{21}$$

Rescaled Risk-Neutral Pricing

by Feynman-Kac, partial differential equation for (15) is below

$$\begin{aligned}\frac{\partial P^{\varepsilon,\delta}}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2 P^{\varepsilon,\delta}}{\partial x^2} + \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}}xf(y)\frac{\partial^2 P^{\varepsilon,\delta}}{\partial x\partial y} + \frac{\nu^2}{\varepsilon}\frac{\partial^2 P^{\varepsilon,\delta}}{\partial y^2} \\ + r\Big(x\frac{\partial P^{\varepsilon,\delta}}{\partial x} - P^{\varepsilon,\delta}\Big) + \Big[\frac{1}{\varepsilon}(m - Y_t^\varepsilon) + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon)\Big]\frac{\partial P^{\varepsilon,\delta}}{\partial y} \\ + \sqrt{\delta}\int\Big(P^{\varepsilon,\delta}(t,x,y,z) - \breve{P}^{\varepsilon,\delta}(t,x,y,z_0)\Big)p(z)dz = 0\end{aligned} \tag{22}$$

which has to be solved for $t < T$ with the terminal condition

$$P^{\varepsilon,\delta}(T,x,y,z) = h(x) \tag{23}$$

Operator Notation

With this notatoin, the pricing partial differential equation becomes

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{A}_{\mathcal{J}}\right) P^{\varepsilon,\delta} = 0 \tag{24}$$

or

$$\mathcal{L}^{\varepsilon,\delta} P^{\varepsilon,\delta} = 0 \tag{25}$$

with terminal conditoin

$$P^{\varepsilon,\delta}(T, x, y, z) = h(x) \tag{26}$$

with

$$\mathcal{L}^{\varepsilon,\delta} := \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{A}_{\mathcal{J}} \quad \text{where} \tag{27}$$

$$\begin{aligned}
\mathcal{L}_0 &:= (m - y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2} \\
\mathcal{L}_1 &:= \sqrt{2}\nu\rho x f(y, z)\frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\Lambda(y, z)\frac{\partial}{\partial y} \\
\mathcal{L}_2 &:= \frac{\partial}{\partial t} + \frac{1}{2} f(y, z)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) \\
\mathcal{A}_{\mathcal{J}} &:= \int \left(\cdot - \breve{\cdot}\right) p(u) du
\end{aligned}$$

Formal Expansion

The method is to expand the solution $P^{\varepsilon,\delta}$ in power of $\sqrt{\delta}$,

$$P^{\varepsilon,\delta} = P_0^\varepsilon + \sqrt{\delta}P_1^\varepsilon + \delta P_2^\varepsilon + \delta\sqrt{\delta}P_3^\varepsilon + \delta^2 P_4^\varepsilon + \cdots \tag{28}$$

where $P_1^\varepsilon, P_2^\varepsilon, ...$ are functin of (t, x, y) to be determined such that $P_0^\varepsilon(T, x, y) = h(x)$.
then

$$\begin{aligned}\mathcal{L}^{\varepsilon,\delta}P^{\varepsilon,\delta} = & \left(\frac{1}{\varepsilon}\mathcal{L}_0 P_0^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 P_0^\varepsilon + \mathcal{L}_2 P_0^\varepsilon\right) \\ & \sqrt{\delta}\left(\frac{1}{\varepsilon}\mathcal{L}_0 P_1^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 P_1^\varepsilon + \mathcal{L}_2 P_1^\varepsilon + \mathcal{A}_{\mathcal{J}} P_0^\varepsilon\right) \\ & \delta\left(\frac{1}{\varepsilon}\mathcal{L}_0 P_2^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 P_2^\varepsilon + \mathcal{L}_2 P_2^\varepsilon + \mathcal{A}_{\mathcal{J}} P_1^\varepsilon\right) \\ & \delta\sqrt{\delta}\left(\frac{1}{\varepsilon}\mathcal{L}_0 P_3^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 P_3^\varepsilon + \mathcal{L}_2 P_3^\varepsilon + \mathcal{A}_{\mathcal{J}} P_2^\varepsilon\right) \\ & + \cdots = 0\end{aligned} \tag{29}$$

The first two coefficient terms

We will use the first two coefficient terms of governing equation.

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}_2\right)P_0^{\varepsilon}=0 \tag{30}$$

with terminal condition $P_0^{\varepsilon}(T,x,y)=h(x)$

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0+\frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1+\mathcal{L}_2\right)P_1^{\varepsilon}=-\mathcal{A}_{\mathcal{J}}P_0^{\varepsilon} \tag{31}$$

with terminal condition $P_1^{\varepsilon}(T,x,y)=0$

Expansion in fast scale P_0^ε

In the first coefficient term, we can also expand P_0^ε as below

$$P_0^\varepsilon = P_0 + \sqrt{\varepsilon} P_{1,0} + \varepsilon P_{2,0} + \varepsilon\sqrt{\varepsilon} P_{3,0} + \varepsilon^2 P_{4,0} + \cdots, \tag{32}$$

then the first coefficient term of governing equation is

$$\begin{aligned}\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_0^\varepsilon =& \frac{1}{\varepsilon}\mathcal{L}_0 P_0 \\ &+ \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0\right) \\ &+ \left(\mathcal{L}_0 P_{2,0} + \mathcal{L}_1 P_{1,0} + \mathcal{L}_2 P_0\right) \\ &+ \sqrt{\varepsilon}(\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0}) \\ &+ \cdots = 0\end{aligned} \tag{33}$$

new coefficient terms

$$\mathcal{L}_0 P_0 = 0 \tag{34}$$

It is ordinary differential equations in y and we therefore take the only solutions that do not depend on y; $P_0 = P_0(t, x, z)$

$$\mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0 = 0 \tag{35}$$

The operator $\mathcal{L}_1$ acts only on the y, by the same reason. It is clear that $P_{1,0} = P_{1,0}(t, x, z)$

new coefficient terms

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_1 P_{1,0} + \mathcal{L}_2 P_0 = 0 \tag{36}$$

by the same reason $\mathcal{L}_1 P_{1,0} = 0$ that means

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_2 P_0 = 0 \tag{37}$$

this equation is poisson equation respects to y. It has the centering solvability condition. therefore $\langle \mathcal{L}_2 P_0 \rangle = 0$ It is the solution of the Black-Scholes equation with mean of volatility $\sigma\bar{(z)}^2 = \langle f(\cdot, z)^2 \rangle$

$$\mathcal{L}_{BS}(\bar{\sigma}(z)^2) P_0 = 0 \tag{38}$$

. where $\langle \mathcal{L}_2 \rangle = \mathcal{L}_{BS}(\bar{\sigma}(z)^2)$ and $\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2}\langle f(\cdot, z)^2 \rangle x^2 \frac{\partial^2}{\partial x^2} + r\left(x \frac{\partial}{\partial x} - \cdot\right)$

new coefficient terms

In the poisson equation

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_2 P_0 = 0 \tag{39}$$

where $\langle \mathcal{L}_2 \rangle = \mathcal{L}_{BS}(\bar{\sigma}(z)^2)$.
The solution of poisson equation $P_{2,0}(t,x)$ is

$$P_{2,0}(t,x,y) = -\,\mathcal{A}P_0 \tag{40}$$

where $\mathcal{A}P_0 = \frac{1}{2}(\phi(y) + c(t,x))x^2 \frac{\partial^2 P_0}{\partial x^2}$ with $\mathcal{L}_0\phi(y) = f(y,z)^2 - \bar{\sigma}(z)^2)$
Because

$$\begin{aligned} \mathcal{L}_0 P_{2,0} &= -\mathcal{L}_2 P_0 = -(\mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle) \\ &= -\frac{1}{2}(f(y,z)^2 - \bar{\sigma}(z)^2)x^2 \frac{\partial^2 P_0}{\partial x^2} \end{aligned} \tag{41}$$

and

$$P_{2,0} = -\frac{1}{2}\mathcal{L}_0^{-1}(f(y,z)^2 - \bar{\sigma}(z)^2)x^2 \frac{\partial^2 P_0}{\partial x^2} \tag{42}$$

new coefficient terms

In the second order differential equation of

$$\mathcal{L}_0\phi(y) = f(y,z)^2 - \bar{\sigma}(z)^2 \tag{43}$$

We know that

$$\phi' = -\frac{1}{\nu^2\Phi}\int_{-\infty}^{\cdot}(f^2 - \langle f^2\rangle)\Phi \tag{44}$$

where $\Phi(y)$ is invariant distribution of y, $\Phi(y) \sim \mathcal{N}(m,\nu^2)$

new coefficient terms

In the poisson equation of third coefficient term

$$\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} = 0 \tag{45}$$

by the same argument,

$$\begin{aligned}\mathcal{L}_{BS}(\bar{\sigma}(z))P_{1,0} &= \frac{1}{2}\langle \mathcal{L}_1 \phi(y)\rangle x^2 \frac{\partial^2 P_0}{\partial x^2} \\ &= \frac{\sqrt{2}}{2}\nu\rho\langle f\phi'\rangle x^3 \frac{\partial^3 P_0}{\partial x^3} + \left(\sqrt{2}\nu\rho\langle f\phi'\rangle - \frac{\sqrt{2}}{2}\nu\langle \Lambda\phi'\rangle\right)\frac{\partial^2 P_0}{\partial x^2}\end{aligned} \tag{46}$$

The Result of Expansion in P_0^ε

In the expansion,

$$P_0^\varepsilon = P_0 + \sqrt{\varepsilon}P_{1,0} + \varepsilon P_{2,0} + \varepsilon\sqrt{\varepsilon}P_{3,0} + \varepsilon^2 P_{4,0} + \cdots, \tag{47}$$

Let $\tilde{P}_{1,0} := \sqrt{\varepsilon}P_{1,0}$ then

$$\mathcal{L}_{BS}(\bar{\sigma}(z))\tilde{P}_{1,0} = \sqrt{\varepsilon}\mathcal{H}(t,x,z)P_0 \tag{48}$$

where $\mathcal{H}(t,x,z) := \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)\rangle$
The solution of this equation is

$$\tilde{P}_{1,0} = -(T-t)\sqrt{\varepsilon}\mathcal{H}(t,x,z)P_0 \tag{49}$$

Expansion in fast scale P_1^ε

We can expand P_1^ε as below

$$P_1^\varepsilon = P_{0,1} + \sqrt{\varepsilon} P_{1,1} + \varepsilon P_{2,1} + \varepsilon\sqrt{\varepsilon} P_{3,1} + \varepsilon^2 P_{4,1} + \cdots, \tag{50}$$

then the second coefficient term of governing equation is

$$\begin{aligned}\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_1^\varepsilon + \mathcal{A}_\mathcal{J} P_0^\varepsilon =& \frac{1}{\varepsilon}\mathcal{L}_0 P_{0,1} + \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_0 P_{1,1} + \mathcal{L}_1 P_{0,1}\right)\\ &+ \left(\mathcal{L}_0 P_{2,1} + \mathcal{L}_1 P_{1,1} + \mathcal{L}_2 P_{0,1} + \mathcal{A}_\mathcal{J} P_0\right)\\ &+ \sqrt{\varepsilon}(\mathcal{L}_0 P_{3,1} + \mathcal{L}_1 P_{2,1} + \mathcal{L}_2 P_{1,1} + \mathcal{A}_\mathcal{J} P_{1,0})\\ &+ \sqrt{\varepsilon}(\mathcal{L}_0 P_{4,1} + \mathcal{L}_1 P_{3,1} + \mathcal{L}_2 P_{2,1} + \mathcal{A}_\mathcal{J} P_{2,0})\\ &+ \cdots = 0\end{aligned} \tag{51}$$

The Result of Expansion in P_1^{ε}

With same argument of expansion in P_0^{ε}, and from the third coefficient term,

$$\langle \mathcal{L}_2 \rangle P_{0,1} = -\langle \mathcal{A}_{\mathcal{J}} \rangle P_0 \tag{52}$$

the solution of this equation is

$$P_{0,1} = -(T-t)\langle \mathcal{A}_{\mathcal{J}} \rangle P_0 \tag{53}$$

because

$$\begin{aligned} \langle \mathcal{L}_2 \rangle P_{0,1} &= \langle \mathcal{L}_2 \rangle \left(-(T-t)\langle \mathcal{A}_{\mathcal{J}} \rangle P_0\right) \\ &= \langle \mathcal{L}_2 \rangle \left(-(T-t)\right) \left(\langle \mathcal{A}_{\mathcal{J}} \rangle P_0\right) - (T-t)\langle \mathcal{L}_2 \rangle \left(\langle \mathcal{A}_{\mathcal{J}} \rangle P_0\right) \\ &= \langle \mathcal{L}_2 \rangle \left(-(T-t)\right) \left(\langle \mathcal{A}_{\mathcal{J}} \rangle P_0\right) - 0 \end{aligned} \tag{54}$$

with the generator of markovian jump $\mathcal{A}_{\mathcal{J}}$

Corrected Price

$$
\begin{aligned}
P^{\varepsilon,\delta} \approx \widetilde{P^{\varepsilon,\delta}} :=& P_0 + \sqrt{\varepsilon}P_{1,0} + \sqrt{\delta}P_{0,1} \\
=& P_0 - (T-t)\sqrt{\varepsilon}\mathcal{H}(t,x)P_0 - \sqrt{\delta}(T-t)\langle \mathcal{A}_{\mathcal{J}}\rangle P_0 \\
=& P_0 - (T-t)\left(\sqrt{\varepsilon}\mathcal{H}(t,x) + \sqrt{\delta}\langle \mathcal{A}_{\mathcal{J}}\rangle\right)P_0 \\
=& P_0 - (T-t)\left(V_2^{\varepsilon}\frac{\partial^2}{\partial x^2} + V_3^{\varepsilon}x^3\frac{\partial^3}{\partial x^3} + V_1^{\delta}\right)P_0
\end{aligned}
\tag{55}
$$

where

$$
\begin{aligned}
V_1^{\delta} &:= \sqrt{\delta}\int \left(\cdot - \breve{\cdot}\right)p(u)du \\
V_2^{\varepsilon} &:= \frac{\nu}{\sqrt{2\alpha}}\left(2\rho\langle f\phi'\rangle - \langle \Lambda\phi'\rangle\right) \\
V_3^{\varepsilon} &:= \frac{\rho\nu}{\sqrt{2\alpha}}\langle f\phi'\rangle
\end{aligned}
\tag{56}
$$

Summary for Option Price of SV With Regimes Shifts

$$\begin{aligned} dX_t =& \mu X_t dt + \sigma_t X_t dB_t^{(1)} \\ \sigma_t =& f(Y_t, Z_t). \\ dY_t =& \alpha(m - Y_t)dt + \beta d\hat{B}_t^{(2)} \\ Z_t :=& \text{Markovian Jump Process} \end{aligned} \tag{57}$$

the corrected price is given explicitly by

$$P^{\varepsilon,\delta} = P_0 - (T-t)\left(\sqrt{\varepsilon}\mathcal{H}(t,x) + \sqrt{\delta}\langle\mathcal{A}_{\mathcal{J}}\rangle\right)P_0 \tag{58}$$

where $P_0 = P_{BS}(\bar{\sigma})$,the Black-Scholes price with constant volatility $\bar{\sigma} = \langle f(y)\rangle$

Compare the Models

$$
\begin{aligned}
P &= P_0 \\
P^{\varepsilon} &= P_0 - (T-t)\sqrt{\varepsilon}\mathcal{H}(t,x)P_0 \\
P^{\varepsilon,\delta} &= P_0 - (T-t)\Big(\sqrt{\varepsilon}\mathcal{H}(t,x) - \frac{\sqrt{\delta}}{2}\langle \mathcal{M}_1 \rangle\Big)P_0 \\
P^{\varepsilon,\delta} &= P_0 - (T-t)\left(\sqrt{\varepsilon}\mathcal{H}(t,x) + \sqrt{\delta}\langle \mathcal{A}_{\mathcal{J}} \rangle\right)P_0
\end{aligned}
\tag{59}
$$

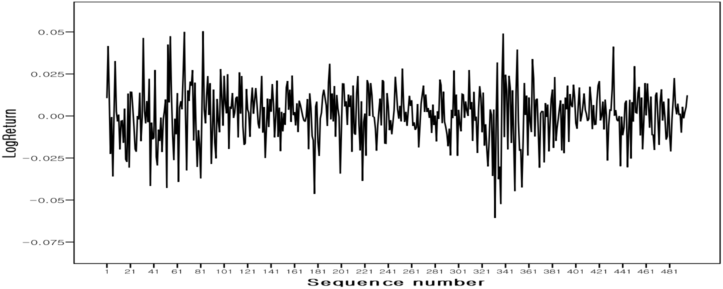
Estimation of Parameter and Simulation

1. Detecting The Regiomes Shifts intensity λ : by Sequential F-test - Rodionov R.(2001) and GMM - Kou(2002))
2. Find out Reverting ration α, β : by Variogram Analysis(same as Periodogram Analysis) - Fouque, et.al (2003)
3. Estimate the μ by GMM - Chan, et.al(1998) and Bollerslev(2001) EMM - Andersen(1997), MCMC- Han(2003)
4. Find out the proxy value of V_2, V_3 : by Regression with LMMR Fouque(1999,2003)

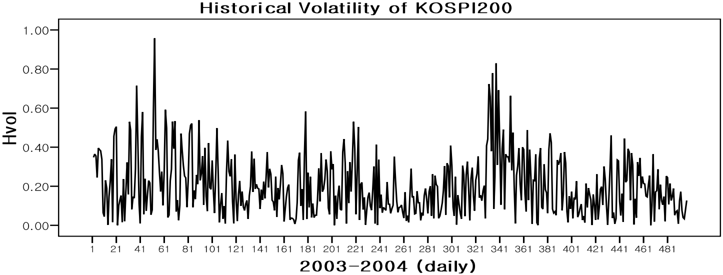
After step3, we can simulate the asset price process with stochastic volatility and markovian regimes shifts and compute the Black-Scholes option price

After step4, we can compute the corrected option price with stochastic volatility and markovian regimes shifts.

Time Series of Log Return 2003-2004

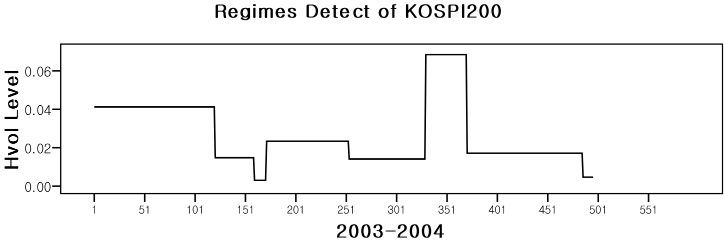


Historical Volatility



Regime Shifts in Markovian Jump process

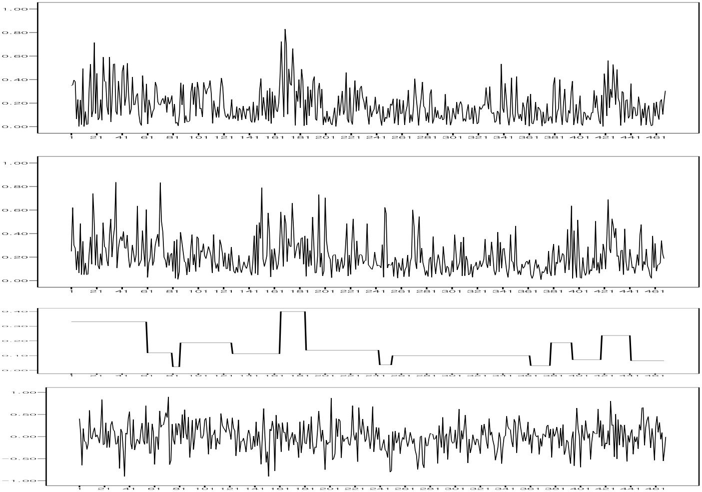
by Sequential F-test - Rodionov R.(2001)



Estimation of Parameter and Simulation

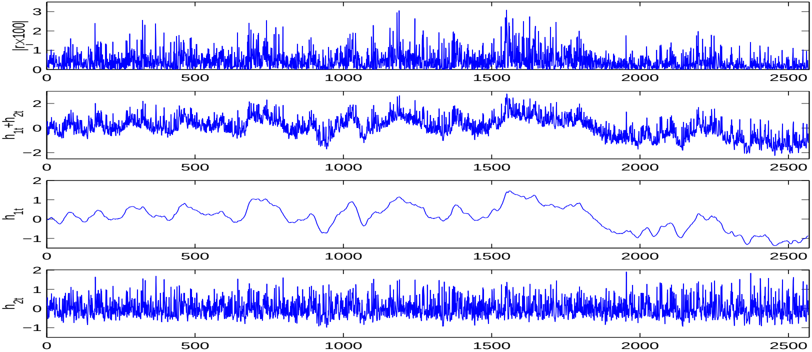
- $\lambda = [0.05, 0.06], \delta = 0.003$
- $\alpha = 200, \varepsilon = 0.05$: 1.5 days, $\beta = 5.2$
- $m = \bar{\sigma} = 0.07$
- mu : not needed replace by interest rate (90 days CD rate)
- $V_2 = -0.0044, V_3 = 0.000154$ from LMMR (Fouque's method)
- $\langle \mathcal{A}_{\mathcal{J}} \rangle$ 0.0001

Simulation of Volatility Process



Simulation of Multiscale Stochastic Volatility

Molina G., Han, C.H., and Fouque J.P.(2004)



Conclusion

Strong point

- Allow regimes shifts in stochastic volatility process
- improved than Fouque and Black-Scholes (in simulation).

Weak point

- complex structure
- do not detect big jump

Further Research

- another jump process to fit the real stochastic volatility process
- easy way to compute the price

Thank you !